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SPIN L-FUNCTIONS ON GSp_8 AND GSp_{10}

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Abstract. The "spin" L-function of an automorphic representation of GSp_{2n} is an Euler product of degree 2^n associated with the spin representation of the L-group GSpin$(2n+1)$. If $n = 4$ or 5, and the automorphic representation is generic in the sense of having a Whittaker model, the analytic properties of these L-functions are studied by the Rankin-Selberg method.

Rankin-Selberg integrals for the spin L-functions associated with generic automorphic cuspidal represenions of certain groups of symplectic similitudes were found by Bump and Ginzburg [B-G]. The groups in question are $GSp(6)$, $GSp(8)$, $GSp(10)$, and $GSp(6) \times GL(2)$. A further Rankin-Selberg integral of this class was treated by Jiang [J], for $GSp(4) \times GSp(4)$. These series of Rankin-Selberg integrals are also analogous to the "spin" integrals on similitude groups of even orthogonal groups, treated in Ginzburg [G1].

Details of the integrals which were announced in [B-G] have not yet appeared in print, except the case of $GSp(6)$ which was treated in more detail by Vo [V]. Vo proved the uniqueness of an associated bilinear form and derived certain consequences, notably the analytic continuation of the local zeta integrals, as well as the nonvanishing results needed to conclude the analyticity of the L-function, except at $s = 0$ and $s = 1$. We treat here the cases of $GSp(8)$ and $GSp(10)$, where we obtain similar results.

It would be desirable to extend this work in two ways. First, note that the restriction to *generic* automorphic cuspidal representations excludes *Siegel modular forms*. It is conjectured that every tempered L-packet on $GSp(2n)$ should contain a generic representative, and if this were known, the construction of the spin L-function for generic cuspidal representations would be *indirectly* applicable to Siegel modular forms. It would be highly desirable to have Rankin-Selberg integrals for the spin L-functions which are valid for Siegel modular forms on $GSp(2n)$. Such integrals are unknown if $2n > 4$.

The second direction of generalization is that we would like to have Rankin-Selberg integrals for spin L-functions on $GSp(2n)$ for *all* n. However we believe (based on case-by-case searching through the possible integrals) that the type of construction considered here (with convolutions unfolding to Whittaker models) stops with $GSp(10)$. There is an intriguing similar construction on $GSp(12)$, but it unfolds to a nonunique model.

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1. Notation

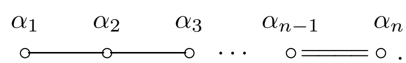
Let $J(n)$ denote the $n \times n$ matrix:

$$J(n) = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

and let

$$J_n = \begin{pmatrix} 0 & J(n) \\ -J(n) & 0 \end{pmatrix}.$$

We define the similitude group of Sp_{2n} as $GSp_{2n} = \{g \in GL_{2n} : tgJ_ng = \mu_n(g)J_n,$ $\mu_n(g)$ a scalar$\}$. $\mu_n(g)$ is called the similitude factor of g. Thus $Sp_{2n} = \{g \in GSp_{2n} : \mu_n(g) = 1\}$. We shall label the simple positive roots of Sp_{2n} as follows:



A given positive root $\alpha = \sum_{i=1}^{n} m_i\alpha_i$ with $m_i \geq 0$ will be denoted by $(m_1m_2\cdots m_n)$ and the one parameter unipotent subgroup corresponding to the root α will be denoted by x_α or $x_\alpha(r)$. Similar notations will be used for negative roots.

Let U_n denote the maximal unipotent subgroup of GSp_{2n} consisting of upper triangular unipotent matrices. The maximal torus of GSp_{2n} is given by

$$\mathrm{diag}(at_1, \cdots, at_n, t_n^{-1}, \cdots, t_1^{-1}) .$$

We shall denote this matrix by $h(at_1, \cdots, at_n)$. We also denote $h(a) = h(a, \cdots, a)$, $h_1(t_1) = h(t_1, 1, \cdots, 1)$, etc.

From now on we shall consider the cases $n = 4, 5$.

An important subgroup of GSp_{10} is

$$H = \{(g_1, g_2) \in GSp_6 \times GSp_4 : \mu_3(g_1) = \mu_2(g_2)\}$$

which we embed in GSp_{10} as follows. If $g_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_6$ and $(g_1, g_2) \in H$, then

$$(g_1, g_2) \hookrightarrow \begin{pmatrix} A & & B \\ & g_2 & \\ C & & D \end{pmatrix} .$$

From now on we shall view the elements of H embedded in GSp_{10} as above. We shall denote the center of GSp_{10} by Z. Notice that Z is a subgroup of H.

In the case of GSp_8 we define

$$H = \{(g_1, g_2) \in GSp_2 \times GSp_2 : \mu_2(g_1) = \mu_2(g_2)\}$$

which is embedded in GSp_8 as

$$(g_1, g_2) \hookrightarrow \begin{pmatrix} g_2 & & & & \\ & a_1 & & b_1 & \\ & & g_2 & & \\ & c_1 & & d_1 & \\ & & & & g_2^* \end{pmatrix}$$

where $g_2^* = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ if $g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As in the GSp_{10} case we denote the center of GSp_8 by Z. Thus Z is a subgroup of H.

2. The global theory

Let F be a global field and $\mathbb{A}$ its ring of adeles. If G is an algebraic group, we denote the F and $\mathbb{A}$ points of G by $G(F)$ and $G(\mathbb{A})$, respectively.

Let π be a cusp form of GSp_{2n}, whose space of representation is V_π. Let ω_π denote the central character of π. In this paper we shall assume that π is generic. To explain this let ψ denote a nontrivial additive character of $F\backslash\mathbb{A}$. Given $u \in U_n$, write $u = x_{\alpha_1}(r_1)\cdots x_{\alpha_n}(r_n)u'$ where $r_i \in \mathbb{A}$ and u' is a product of the other positive roots in any fixed order. Define a character $\psi_{\mathbb{A}}$ of $U_n(F)\backslash U_n(\mathbb{A})$ by

$$\psi_{\mathbb{A}}(u) = \psi\left(\sum_{i=1}^{n} r_i\right) .$$

Thus to say that π is generic means that the space of functions generated by

$$W_\varphi(g) = \int\limits_{U_n(F)\backslash U_n(\mathbb{A})} \varphi(ug)\,\psi_{\mathbb{A}}(u)\,du$$

is not identically zero for all choices of $\varphi \in V_\pi$ and $g \in GSp_{2n}(\mathbb{A})$. We shall denote the above space of functions by $\mathcal{W}(\pi,\psi)$. We start with the GSp_{10} case. First we define the Eisenstein series we use. We recall its construction from [G1]. Let $P = MR$ denote the Siegel parabolic subgroup of GSp_6. Thus $M = GL_1 \times GL_3$ which we shall embed in GSp_6 as

$$(\alpha, g) \hookrightarrow \begin{pmatrix} \alpha g & \\ & g^* \end{pmatrix}, \quad (\alpha, g) \in GL_1 \times GL_3,$$

and where g^* is such that the above matrix is in GSp_6. R can be identified with $\{Y \in M_3 : {}^tYJ(3) - J(3)Y = 0\}$ where M_3 is the set of all 3×3 matrices. The identification is

$$Y \hookrightarrow \begin{pmatrix} I_3 & Y \\ & I_3 \end{pmatrix}$$

where I_3 is the 3×3 identity matrix.

Let χ be a unitary character of $F^*\backslash\mathbb{A}^*$ and for $(\alpha, g) \in GL_1(\mathbb{A}) \times GL_3(\mathbb{A})$, set

$$\chi_\pi\big((\alpha, g)\big) = (\omega_\pi\chi^3)(\alpha)\,(\omega_\pi\chi^2)(\det g) .$$

We extend χ_π to a character of P by letting it act trivially on $R(\mathbb{A})$. For $s \in \mathbb{C}$ set $I(s,\chi_\pi) = \mathrm{Ind}_{P(\mathbb{A})}^{GSp_6(\mathbb{A})}\delta_P^s\chi_\pi$ where δ_P denotes the modular function of P. Finally given $f_s \in I(s,\chi_\pi)$ we let

$$E(g, f_s, \chi, s) = \sum_{\gamma \in P(F)\backslash GSp_6(F)} f_s(\gamma g), \quad g \in GSp_6(\mathbb{A}) .$$

Let w denote the Weyl element in GSp_{10} which has 1 at positions (1,1); (2,4); (3,2); (4,5); (5,8); (7,6); (8,9); (9,7); (10,10) and -1 at position (6,3). We set $j(g) = wgw^{-1}$ for $g \in GSp_{10}$.

We are now ready to introduce our global integral. For $\varphi \in V_\pi$, etc. we set

$$I(\varphi,\chi,f_s,s) = \int\limits_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \varphi\big(j(g_1,g_2)\big)\, E(g_1,f_s,\chi,s)\, dg_1\, dg_2\ .$$

Let V denote the maximal unipotent subgroup of H consisting of upper triangular matrices. Our main result in this section is:

Theorem 2.1. *The integral $I(\varphi,\chi,f_s,s)$ converges absolutely for all s where the Eisenstein series $E(g,f_s,\chi,s)$ is holomorphic. For $Re(s)$ large, we have*

$$I(\varphi,\chi,f_s,s) = \int\limits_{Z(\mathbb{A})V(\mathbb{A})\backslash H(\mathbb{A})} \int\limits_{\mathbb{A}} W_\varphi\big(x_{-00001}(r) j(g_1,g_2)\big)\, f_s(g_1)\, dr\, dg_1\, dg_2\ .$$

Proof. The convergence of $I(\varphi,\chi,f_s,s)$ follows easily from the cuspidality of φ. We shall now carry out the unfolding process. The convergence justification of each step is done as in [G1]. We shall use the following notation. Let H_1 be a subgroup of GSp_6 and H_2 a subgroup of GSp_4. Set $(H_1,H_2) = \{(h_1,h_2) \in H_1 \times H_2 : \mu_3(h_1) = \mu_2(h_2)\}$. Thus (H_1,H_2) is a subgroup of H.

We start by unfolding the Eisenstein series. Thus for $Re(s)$ large

$$I(\varphi,\chi,f_s,s) = \int\limits_{Z(\mathbb{A})(P(F),GSp_4(F))\backslash H(\mathbb{A})} \varphi\big(j(g_1,g_2)\big)\, f_s(g_1)\, dg_1\, dg_2\ .$$

Writing $P = MR$ and pulling out the adelic part of R we obtain

$$I(\varphi,\chi,f_s,s) = \int \int\limits_{R(F)\backslash R(\mathbb{A})} \varphi\big(j(rg_1,g_2)\big)\, f_s(g_1)\, dr\, dg_1\, dg_2\ . \tag{2.1}$$

Here (g_1,g_2) are integrated over $Z(\mathbb{A})\big(M(F)R(\mathbb{A}),GSp_4(F)\big)\backslash H(\mathbb{A})$.

Let $P_1 = M_1U_1$ denote the maximal parabolic of GSp_{10} whose Levi part contains the group $GL_3 \times Sp_4$ and such that $U_1 \subset U$. (Recall that U is the maximal unipotent of GSp_{10} consisting of upper triangular matrices.) The group R, embedded in GSp_{10}, is the center of U_1 and if $X = R\backslash U_1$, then $X \simeq M_{3\times 4}$ the group of all 3×4 matrices. Consider in (2.1) the Fourier expansion with respect to $X(F)\backslash X(\mathbb{A})$. Thus

$$I(\varphi,\chi,f_s,s) = \int \sum_\mu \int\limits_{(XR)(F)\backslash(XR)(\mathbb{A})} \varphi\Big(j\big(x(rg_1,g_2)\big)\Big)\, \mu(x)\, dx\, dr\, f_s(g_1)\, dg_1\, dg_2$$

where the summation on μ is over all characters of $X(F)\backslash X(\mathbb{A})$. Recall that $M \simeq GL_1 \times GL_3$. Identifying the group character of $X(F)\backslash X(\mathbb{A})$ with $X(F)$ and X with $M_{3\times 4}$, the group $\big(M(F),GSp_4(F)\big) \simeq \big(GL_1(F)\times GL_3(F),GSp_4(F)\big)$ acts on $X(F)$ as $\alpha z_1 x z_2^{-1}$ where $\big((\alpha_1,z_1),z_2\big) \in \big(GL_1(F)\times GL_3(F),\, GSp_4(F)\big)$. This action has five orbits. Recalling that $X \simeq M_{3\times 4}$, we may choose as representatives

$$x_1 = 0;\quad x_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix};\quad x_3 = \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix};$$

$$x_4 = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix};\quad x_5 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\ .$$

Write $x = (x_{ij})$ with $1 \le i \le 3$ and $1 \le j \le 4$. Define the following characters of $X(\mathbb{A})$. First $\mu_1(x) = 1$; $\mu_2(x) = \psi(x_{31})$; $\mu_3(x) = \psi(x_{21}+x_{32})$; $\mu_4(x) = \psi(x_{22}+x_{33})$ and $\mu_5(x) = \psi(x_{11}+x_{22}+x_{33})$. Denote by $\mathrm{Stab}(x_i)$ the stabilizer of x_i in (M, GSp_4) under the action described above. Thus,

$$I(\varphi,\chi,f_s,s) = \sum_{i=1}^{5} \int \int_{U_1(F)\backslash U_1(\mathbb{A})} \varphi\Big(j\big(u_1(g_1,g_2)\big)\Big)\, \mu_i(u_1)\, du_1\, f_s(g_1)\, dg_1\, dg_2$$

where (g_1, g_2) is integrated over $Z(\mathbb{A})\mathrm{Stab}(x_i)(F)\big(R(\mathbb{A}),1\big)\backslash H(\mathbb{A})$ and μ_i is extended to a character of U_1 by letting it act trivially on R. We claim that only the integral corresponding to x_5 in the above sum is nonzero. To show that the other four contribute zero we shall exhibit in each case a unipotent radical subgroup of U on which μ_i will be trivial. Thus by cuspidality of φ the integral will vanish. For x_1 this is clear since U_1 is a radical of GSp_{10} and $\mu_1 \equiv 1$. For x_2 one can easily check that the stabilizer of x_2 in $M(F)$ contains the unipotent subgroup of GL_3 of elements of the form $\begin{pmatrix} 1 & 0 & r_1 \\ & 1 & r_2 \\ & & 1 \end{pmatrix}$. Thus combining the embedding of this group in GSp_{10} with U_1, one can check that we obtain the unipotent radical of the maximal parabolic in GSp_{10} which preserves a plane. Also $\mu_2 \equiv 1$ on this radical; hence by cuspidality we get zero. Finally in the cases of x_3 and x_4 one can check that we end up integrating φ along the radical of the maximal parabolic which preserves a line and that μ_3 and μ_4 are trivial on this radical. Thus the only contribution to $I(\varphi,\chi,f_s,s)$ comes from the x_5 orbit. The stabilizer of x_5 in (M, GSp_4) is

(2.2)
$$Z\left(\begin{pmatrix} |g| & & \\ & g & & \\ & & g^* & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & z_1 & z_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -z_2 \\ & & & & 1 & -z_1 \\ & & & & & 1 \end{pmatrix}, \begin{pmatrix} |g| & & \\ & g & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & z_1 & z_2 & z_3 \\ & 1 & & z_2 \\ & & 1 & -z_1 \\ & & & 1 \end{pmatrix}\right)$$

where $g \in GL_2$ and $|g| = \det\,(g)$. Here we identified M with $GL_1 \times GL_3$. Denote $\mu_5(u_1) = \psi_{U_1}(u_1)$. Thus

$$I(\varphi,\chi,f_s,s) = \int \varphi\Big(j\big(u_1(g_1,g_2)\big)\Big)\psi_{U_1}(u_1)\, f_s(g_1)\, du_1\, dg_1\, dg_2$$

where u_1 is integrated over $U_1(F)\backslash U_1(\mathbb{A})$ and (g_1, g_2) over $Z(\mathbb{A})\mathrm{Stab}(x_5)(F)\backslash H(\mathbb{A})$. Define

$$V_1 = \left(\begin{pmatrix} 1 & z_1 & z_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -z_2 \\ & & & & 1 & -z_1 \\ & & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & z_1 & z_2 & z_3 \\ & 1 & & z_2 \\ & & 1 & -z_1 \\ & & & 1 \end{pmatrix}\right) \subset H$$

and

$$V_2 = \left(\begin{pmatrix} 1 & y_1 & y_2 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & -y_2 \\ & & & & 1 & -y_1 \\ & & & & & 1 \end{pmatrix}, \quad 1 \right) \subset H.$$

We identify $\mathrm{Stab}(x_5)$ with ZGL_2V_1 in the obvious notation. Clearly $V_2(\mathbb{A})V_1(F) \supset V_1(F)$. Hence we may factorize the integral

$$\int\limits_{Z(\mathbb{A})GL_2(F)V_1(F)(R(\mathbb{A}),1)\backslash H(\mathbb{A})} = \int\limits_{Z(\mathbb{A})GL_2(F)V_2(\mathbb{A})V_1(F)(R(\mathbb{A}),1)\backslash H(\mathbb{A})} \ \int\limits_{V_2(\mathbb{A})}$$

where we have replaced an integral over $V_1(F)\backslash V_2(\mathbb{A})V_1(F)$ with $V_2(\mathbb{A})$. This is justified since $V_1 \cap V_2 = 1$. Also GL_2 is viewed as a subgroup of H as described in (2.2). Thus

$$\begin{aligned} &I(\varphi, \chi, f_s, s) \\ (2.3) \qquad &= \int \int\limits_{V_2(\mathbb{A})} \int\limits_{U_1(F)\backslash U_1(\mathbb{A})} \varphi\Big(j\big(u_1(v_2g_1, g_2)\big)\Big)\, \psi_{U_1}(u_1)\, f_s(g_1)\, du_1\, dv_2\, dg_1\, dg_2 \end{aligned}$$

where (g_1, g_2) is integrated over $Z(\mathbb{A})\big(R(\mathbb{A}), 1\big)GL_2(F)V_2(\mathbb{A})V_1(F)\backslash H(\mathbb{A})$. Let us express the groups V_2 and U_1 in terms of roots in GSp_{10}. First U_1 consists of all positive roots $\alpha = \sum\limits_{i=1}^{5} n_i\alpha_i$ with $n_3 > 0$. The character ψ_{U_1} is nontrivial on the roots (11100); (01110) and (00111). More precisely, write

$$u_1 = x_{11100}(r_1)x_{011100}(r_2)x_{00111}(r_3)u_1'$$

where $u_1' \in U_1$ is a product of all other roots in U_1 in any fixed order. Then

$$(2.4) \qquad \psi_{U_1}(u_1) = \psi(r_1 + r_2 + r_3)\ .$$

The roots in V_2 are (10000) and (11000). Write $u_1 = x_{00100}(z_1)x_{01100}(z_2)u_1'$. Then, for $h \in GSp_{10}(\mathbb{A})$,

(2.5)

$$\begin{aligned} &\int\limits_{V_2(\mathbb{A})} \int\limits_{U_1(F)\backslash U_1(\mathbb{A})} \varphi\Big(j\big(u_1(v_2, 1)h\big)\Big)\psi_{U_1}(u_1)du_1dv_2 \\ &= \int\limits_{\mathbb{A}^2} \int\limits_{(F\backslash \mathbb{A})^2} \int\limits_{U_1'(F)\backslash U_1'(\mathbb{A})} \varphi\Big(j\big(x_{00100}(z_1)x_{01100}(z_2)u_1'\big(x_{10000}(y_1)x_{11000}(y_2), 1\big)h\big)\Big) \\ &\qquad \times \psi_{U_1}(u_1')\, du_1'\, dz_i\, dy_i\ . \end{aligned}$$

Here U_1' is the subgroup of U_1 consisting of all roots in U_1 excluding (00100) and (01100). Also from (2.4) it follows that $\psi_{U_1}(u_1') = \psi_{U_1}(u_1)$. Write $\int\limits_{\mathbb{A}^2} = \int\limits_{(F\backslash\mathbb{A})^2} \sum\limits_{\xi_1,\xi_2\in F}$. It is easy to check that x_{10000} and x_{11000} normalize U_1', and that

$$\begin{aligned} &x_{00100}(z_1)\, x_{01100}(z_2)\, x_{10000}(\xi_2)\, x_{11000}(\xi_1) \\ &\qquad = x_{10000}(\xi_2)\, x_{11000}(\xi_1)\, x_{00100}(z_1)\, x_{01100}\,(z_2)\, x_{11100}(\xi_1 z_1 + \xi_2 z_2)\ . \end{aligned}$$

Thus (2.5) equals

$$\int\limits_{(F\backslash\mathbb{A})^4}\sum_{\xi_1,\xi_2\in F}\int\limits_{U_1'(F)\backslash U_1'(\mathbb{A})}\varphi\Big[j\big(x_{00100}(z_1)x_{01100}(z_2)u_1'\big(x_{10000}(y_1)x_{11000}(y_2),1\big)h\big)\Big]$$
$$\times\,\psi_{U_1}(u_1')\,\psi(\xi_1z_1+\xi_2z_2)\,du_1'\,dz_i\,dy_i\ .$$

Here we used the left invariance property of φ under rational points and we also used change of variables in z_1,z_2 and u_1'. Using the Fourier inversion formula this equals

$$\int\limits_{(F\backslash\mathbb{A})^2}\ \int\limits_{U_1'(F)\backslash U_1'(\mathbb{A})}\varphi\Big[j\big(u_1'\big(x_{10000}(y_1)x_{11000}(y_2),1\big)h\big)\Big]\,\psi_{U_1}(u_1')\,du_1'\,dy_i\ .$$

Write in (2.3)

$$\int\limits_{Z(\mathbb{A})(R(\mathbb{A}),1)GL_2(F)V_2(\mathbb{A})V_1(F)\backslash H(\mathbb{A})}=\int\limits_{Z(\mathbb{A})(R(\mathbb{A}),1)GL_2(F)V_2(\mathbb{A})V_1(\mathbb{A})\backslash H(\mathbb{A})}\ \int\limits_{V_1(F)\backslash V_1(\mathbb{A})}\ .$$

Define $V_3=V_1V_2(R,1)$ and $U_2=V_3U_1'$ (here as before we view V_1,V_2 and V_3 as unipotent subgroups of GSp_{10}). Thus U_2 consists of all roots in U_1' and V_2, including the positive roots (00010); (00011) and (00021). Extend the character ψ_{U_1} from U_1' to U_2 trivially. This is well defined. Thus we obtain

$$I(\varphi,\chi,f_s,s)=\int\ \int\limits_{U_2(F)\backslash U_2(\mathbb{A})}\varphi\Big[j\big(u_2(g_1,g_2)\big)\Big]\,\psi_{U_2}(u_2)\,f_s(g_1)\,du_2\,dg_1\,dg_2$$

where (g_1,g_2) are integrated over $Z(\mathbb{A})V_3(\mathbb{A})GL_2(F)\backslash H(\mathbb{A})$. Next, in the above integral, we consider the Fourier expansion with respect to the roots $x_{-011000}(y_1)$ and $x_{-00100}(y_2)$ where $y_1,y_2\in F\backslash\mathbb{A}$. In a similar way as before we see that the $GL_2(F)$ (as embedded in GSp_{10} as above) acts on the character group of the unipotent group generated by the above two negative roots. The action is via the standard representation of GL_2 and hence there are two orbits. The trivial orbit contributes zero since we may recover, as a normal subgroup, a unipotent radical which is a conjugate to the radical of the standard parabolic subgroup of GSp_{10} whose Levi part contains $GL_2\times Sp_6$. As for the second orbit define the character

$$\widetilde{\psi}\big(x_{-01100}(y_1)x_{-00100}(y_2)\big)=\psi(y_1)\ .$$

We may choose $\widetilde{\psi}$ as a representative of the open orbit. The stabilizer of $\widetilde{\psi}$ in $GL_2(F)$ is the group L of matrices of the form $\begin{pmatrix}\alpha&\beta\\0&1\end{pmatrix}$, $\alpha\in F^*$, $\beta\in F$. Define U_3 to be the unipotent subgroup of GSp_{10} generated by the roots in U_2 and the roots $-(01100)$ and $-(00100)$. For $u_3=x_{-01100}(y_1)x_{-00100}(y_2)u_2$ set

$$\psi_{U_3}(u_3)=\psi(y_1)\psi_{U_2}(u_2)\ .$$

Then

$$I(\varphi,\chi,f_s,s)=\int\ \int\limits_{U_3(F)\backslash U_3(\mathbb{A})}\varphi\Big[j\big(u_3(g_1,g_2)\big)\Big]\,\psi_{U_3}(u_3)\,f_s(g_1)\,du_3\,dg_1\,dg_2 \tag{2.6}$$

where (g_1, g_2) are integrated over $Z(\mathbb{A})V_3(\mathbb{A})L(F)\backslash H(\mathbb{A})$. The group L of matrices of the form $\begin{pmatrix}\alpha & z\\ & 1\end{pmatrix}$ is embedded in H as follows:

$$\left(\begin{pmatrix}\alpha &&&&&\\ &\alpha & z &&&\\ &&1&&&\\ &&&\alpha&-z&\\ &&&&1&\\ &&&&&1\end{pmatrix}, \begin{pmatrix}\alpha &&&\\ &\alpha&z&\\ &&1&\\ &&&1\end{pmatrix}\right). \tag{2.7}$$

Next we carry out a process similar to that performed after (2.2) and in (2.5). Namely, define

$$V_4 = \left(\begin{pmatrix}1 &&&&&\\ &1 & z &&&\\ &&1&&&\\ &&&1&-z&\\ &&&&1&\\ &&&&&1\end{pmatrix}, \begin{pmatrix}1 &&&\\ &1&z&\\ &&1&\\ &&&1\end{pmatrix}\right)$$

and

$$V_5 = \left(\begin{pmatrix}1 &&&&&\\ &1 & z &&&\\ &&1&&&\\ &&&1&-z&\\ &&&&1&\\ &&&&&1\end{pmatrix}, 1\right).$$

We have $V_5(\mathbb{A})V_4(F) \supset V_4(F)$ and hence

$$\int\limits_{Z(\mathbb{A})V_3(\mathbb{A})GL_1(F)V_4(F)\backslash H(\mathbb{A})} = \int\limits_{Z(\mathbb{A})V_3(\mathbb{A})V_5(\mathbb{A})V_4(F)GL_1(F)\backslash H(\mathbb{A})} \int\limits_{V_5(\mathbb{A})}.$$

Let U_3' denote the subgroup of U_3 excluding the root (00110). Thus, for $u_3 = x_{00110}(y)u_3'$

$$\int\limits_{\mathbb{A}} \int\limits_{U_3(F)\backslash U_3(\mathbb{A})} \varphi\Big[j\big(u_3\big(x_{01000}(z)g_1, g_2\big)\big)\Big]\psi_{U_3}(u_3)du_3dz$$
$$= \sum_{\xi\in F} \int\limits_{(F\backslash\mathbb{A})^2} \int\limits_{U_3'(F)\backslash U_3'(\mathbb{A})} \varphi\Big[j\big(x_{00110}(y)u_3'\big(x_{01000}(z+\xi)g_1, g_2\big)\big)\Big]\psi_{U_3}(u_3')du_3'dydz\,.$$

We have

$$x_{00110}(y)x_{01000}(\xi) = x_{01000}(\xi)x_{00110}(y)x_{01110}(\xi y)$$

and also 01000 normalizes U_3'. Thus after a change of variables, we get

$$\sum_{\xi} \int\limits_{(F\backslash\mathbb{A})^2} \int\limits_{U_3'(F)\backslash U_3'(\mathbb{A})} \varphi\Big[j\big(x_{00110}(y)u_3'\big(x_{01000}(z)g_1, g_2\big)\big)\Big]\,\psi_{U_3}(u_3')\,\psi(\xi y)\,du_3'\,dy\,dz$$

which, by the Fourier inversion formula, equals

$$\int_{F\backslash\mathbb{A}} \int_{U_3'(F)\backslash U_3'(\mathbb{A})} \varphi\Big[j\big(u_3'(x_{01000}(z)g_1, g_2)\big)\Big]\, \psi_{U_3}(u_3')\, du_3'\, dz\ .$$

Let U_4 denote the unipotent subgroup of GSp_{10} which consists of the roots in U_3' including the roots (01000) and (00001). Notice also that $V = V_3V_4V_5$ (where V is the maximal unipotent of H). Thus (2.6) equals

$$\int \int_{U_4(F)\backslash U_4(\mathbb{A})} \varphi\Big[j\big(u_4(g_1, g_2)\big)\Big]\, \psi_{U_4}(u_4)\, f_s(g_1)\, du_4\, dg_1\, dg_2$$

where (g_1, g_2) are integrated over $Z(\mathbb{A})GL_1(F)V(\mathbb{A})\backslash H(\mathbb{A})$ and

$$\psi_{U_4}(u_4) = \psi_{U_4}\big(x_{01000}(y_1)x_{00001}(y_2)u_3'\big) = \psi_{U_3}(u_3').$$

Next we consider a Fourier expansion with respect to $x_{-00110}(y)$ with $y \in F\backslash\mathbb{A}$. Thus

$$I(\varphi, \chi, f_s, s) = \sum_{\xi} \int \int_{F\backslash\mathbb{A}} \int_{U_4(F)\backslash U_4(\mathbb{A})} \varphi\Big[j\big(x_{-00110}(y)u_4(g_1, g_2)\big)\Big] \\ \times \psi_{U_4}(u_4)\, \psi(\xi y)\, f_s(g_1)\, du_4\, dy\, dg_1\, dg_2\ .$$

Let U_4' denote the subgroup of U_4 obtained by excluding the root (00221). Write $u_4 = u_4'x_{00221}(z)$. We have, for $\xi \in F$,

$$x_{00221}(\xi)x_{-00110}(y) = x_{-00110}(y)x_{00221}(\xi)x_{00111}(\xi y)x_{00001}(r)$$

where $r \in F$. Substituting this into the above integral we obtain

$$I(\varphi, \chi, f_s, s) = \sum_{\xi} \int \int_{(F\backslash\mathbb{A})^2} \int_{U_4'(F)\backslash U_4'(\mathbb{A})} \varphi\Big[j\big(x_{-00110}(y)u_4'x_{00221}(z+\xi)(g_1, g_2)\big)\Big] \\ \times \psi_{U_4}(u_4')\, f_s(g_1)\, du_4'\, dy\, dz\, dg_1\, dg_2\ .$$

Here we used the left invariance property of φ by $x_{00221}(\xi)$. We also used a change of variables in u_4' which caused the factor $\psi(\xi y)$ to cancel since ψ_{U_4} is nontrivial on the root (00111). Thus

$$I(\varphi, \chi, f_s, s) \\ = \int \int_{\mathbb{A}} \int_{U_5(F)\backslash U_5(\mathbb{A})} \varphi\Big[j\big(u_5x_{00221}(z)(g_1, g_2)\big)\Big]\, \psi_{U_5}(u_5)\, f_s(g_1)\, du_5\, dz\, dg_1\, dg_2\ .$$

Here U_5 is the unipotent subgroup of GSp_{10} consisting of the roots in U_4' including $-(00110)$, and

$$\psi_{U_5}(u_5) = \psi_{U_5}\big(x_{-(00110)}(y)u_4'\big) = \psi_{U_4}(u_4')\ .$$

Finally we consider the expansion with respect to $x_{-00221}(y)$ with $y \in F\backslash\mathbb{A}$. The group GL_1, embedded in GSp_1 as in (2.7), acts on the characters of this unipotent group with two orbits. The trivial one contributes zero by cuspidality and the open orbit implies the theorem. □

Next we consider the GSp_8 case. Let χ be a unitary character of $F^*\backslash\mathbb{A}^*$. Denote by B the standard Borel subgroup of GL_2. If π is a cusp form of GSp_8 with central character ω_π, let χ_π denote the character of B defined by

$$\chi_\pi \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \omega_\pi \chi(a)\chi^{-1}(b) \ .$$

Set $I(s,\chi_\pi) = \mathrm{Ind}_{B(\mathbb{A})}^{GL_2(\mathbb{A})} \delta_B^s \chi_\pi$ where δ_B is the modular function of B. For $f_s \in I(s,\chi_\pi)$ we let

$$E(g,f_s,\chi,s) = \sum_{\gamma \in B(F)\backslash GL_2(F)} f_s(\gamma g), \quad g \in GL_2(\mathbb{A}) \ .$$

Let R be the unipotent radical of GSp_8 which consists of all positive roots of the form $\sum_{i=1}^{4} n_i\alpha_i$ with $n_2 \geq 1$. If $r = x_{1110}(\ell_1)x_{0111}(\ell_2)r'$ where r' runs over all other roots in R, define $\psi_R(r) = \psi(\ell_1+\ell_2)$. Let w denote the Weyl element of GSp_8 which has 1 at positions (1,3); (2,1); (3,4); (4,7); (6,5); (7,8); (8,6) and -1 at (5,2). Set $j(g) = wgw^{-1}$ for $g \in GSp_8$.

We define

$$J(\varphi,\chi,f_s,s) = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \int_{R(F)\backslash R(\mathbb{A})} \varphi\big(j(r(g_1,g_2))\big)\, \psi_R(r)\, E(g_1,f_s,\chi,s)\, dr\, dg_1\, dg_2$$

where $\varphi \in V_\pi$ is a cusp form of $GSp_8(\mathbb{A})$. Let V denote the maximal unipotent subgroup of H consisting of upper triangular matrices. We prove:

Theorem 2.2. *The integral $J(\varphi,\chi,f_s,s)$ converges absolutely for all s for which the Eisenstein series is entire. For $Re(s)$ large,*

(2.8)
$$J(\varphi,\chi,f_s,s) = \int_{Z(\mathbb{A})V(\mathbb{A})\backslash H(\mathbb{A})} \int_{\mathbb{A}^4} W_\varphi\Big(X(m_1,m_2,m_3,m_4)\, j\big((g_1,g_2)\big)\Big)\, f_s(g_1)\, dm_i\, dg_1\, dg_2$$

where $X(m_1,m_2,m_3,m_4) = x_{-1000}(m_1)x_{-1111}(m_2)x_{-0011}(m_3)x_{-0001}(m_4)$.

Proof. Unfolding the Eisenstein series we obtain for $Re(s)$ large,

$$J(\varphi,\chi,f_s,s) = \int_{Z(\mathbb{A})L(\mathbb{A})GL_2^{\Delta}(F)\backslash H(\mathbb{A})} \int_{R(F)\backslash R(\mathbb{A})} \int_{F\backslash\mathbb{A}} \varphi\big(j(x_{0012}(m)r(g_1,g_2))\big) \times \psi_R(r)\, f_s(g_1)\, dm\, dr\, dg_1\, dg_2$$

where GL_2^{Δ} is the group of matrices of the form $\left(\begin{pmatrix} \det g & \\ & 1 \end{pmatrix}, g\right) \subset H$, and L is the group of matrices of the form $\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, 1\right) \subset H$.

Consider the Fourier expansion along $x_{0010}(t_1)x_{0011}(t_2)$. Thus

$$J(\varphi,\chi,f_s,s)=\int\sum_{\xi_1,\xi_2\in F}\int_{(F\backslash\mathbb{A})^2}\varphi\big(j(x_{0010}(t_1)x_{0011}(t_2)x_{0012}(m)r(g_1,g_2))\big)$$
$$\times\,\psi_R(r)\,\psi(\xi_1t_1+\xi_2t_2)\,f_s(g_1)\,dt_i\,dm\,dr\,dg_1\,dg_2\ .$$

Using the left invariance property of φ with respect to the rational point we conjugate the matrix $j\big(x_{1100}(\xi_1)x_{0100}(\xi_2)\big)$ from left to right. Changing variables and collapsing the summation over ξ_1,ξ_2 with the suitable integration we obtain

$$J(\varphi,\chi,f_s,s)=\int\int_{R_1(F)\backslash R_1(\mathbb{A})}\int_{\mathbb{A}^2}\varphi\big(j(r_1x_{1100}(m_1)x_{0100}(m_2)(g_1,g_2))\big)$$
$$\times\,\psi_{R_1}(r_1)\,f_s(g_1)\,dr_1\,dm_1\,dm_2\,dg_1\,dg_2\ .$$

Here $R_1\subset U_4$ is the radical of the parabolic subgroup of GSp_8 whose Levi part is $GL_3\times GL_2$. Thus R_1 consists of all positive roots of GSp_8 of the form $\sum_{i=1}^{4}n_i\alpha_i$ with $n_3\geq 1$. Also if $r_1=x_{1110}(\ell_1)x_{0111}(\ell_2)r_1'$, then $\psi_{R_1}(r_1)=\psi(\ell_1+\ell_2)$.

Next we consider the Fourier expansion along the roots $x_{-1100}(t_1)x_{-0100}(t_2)$ with $t_i\in F\backslash\mathbb{A}$. The group $GL_2^{\Delta}(F)$ acts on the group character of these two negative roots with two orbits. It is not hard to check that the trivial orbit contributes zero by cuspidality of φ. Thus

$$J(\varphi,\chi,f_s,s)$$
$$=\int_{Z(\mathbb{A})P(F)L(\mathbb{A})\backslash H(\mathbb{A})}\int_{R_2(F)\backslash R_2(\mathbb{A})}\int_{\mathbb{A}^2}\varphi\big(j(r_2x_{1100}(m_1)x_{0100}(m_2)(g_1,g_2))\big)$$
$$\times\,\psi_{R_2}(r_2)\,f_s(g_1)\,dm_i\,dr_2\,dg_1\,dg_2\ .$$

Here R_2 consists of all roots in R_1 including the roots x_{-1100} and x_{-0100}. If $r_2=x_{-1100}(\ell_1)x_{1110}(\ell_2)x_{0111}(\ell_3)r_2'$, then $\psi_{R_2}(r_2)=\psi(\ell_1+\ell_2+\ell_3)$. With this definition we have $P=\left(\begin{pmatrix}a&\\&1\end{pmatrix},\begin{pmatrix}a&b\\0&1\end{pmatrix}\right)$. We repeat the first step of the unfolding. First we expand along the root x_{1000} and use the root x_{0110}, and then we expand along x_{-0110} and use the root x_{0122} to obtain

$$J(\varphi,\chi,f_s,s)=\int_{Z(\mathbb{A})GL_1(F)V(\mathbb{A})\backslash H(\mathbb{A})}\int_{R_3(F)\backslash R_3(\mathbb{A})}\int_{\mathbb{A}^4}\varphi\big(j(r_3x_{1100}(m_1)x_{0100}(m_2)$$
$$\times\,x_{1000}(m_3)x_{0122}(m_4)(g_1,g_2))\big)\psi_{R_3}(r_3)\,f_s(g_1)\,dm_i\,dr_3\,dg_1\,dg_2\ .$$

Here R_3 consists of all roots in R_2 except x_{0110} and x_{0122} and including the roots x_{1000}, x_{0001}, x_{-0110}. Thus dim $R_3=15$. Also ψ_{R_3} is nontrivial on the roots $x_{-1100},x_{1110},x_{0111}$. Finally GL_1 is the group of $\left(\begin{pmatrix}a&\\&1\end{pmatrix},\begin{pmatrix}a&\\&1\end{pmatrix}\right)\subset H$. To finish the unfolding process we expand along $x_{-0122}(t)$ with $t\in F\backslash\mathbb{A}$. $GL_1(F)$ acts on the character group of this root with two orbits. The trivial orbit contributes zero by cuspidality and the open orbit implies (2.8). Indeed one can check that

$$j\big(x_{1100}(m_1)x_{0100}(m_2)x_{1000}(m_3)x_{0122}(m_4)\big)=X(m_1,m_2,m_3,m_4)\ .$$

□

It follows from Theorem 2.1 that $I(\varphi,\chi,f_s,s)$ is factorizable. Indeed, let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation, and $\chi = \bigotimes_v \chi_v$ a unitary character. Let $I(s,\chi_\pi) = \bigotimes_v I(s,\chi_v)$. Assuming $\varphi = \bigotimes_v \varphi_v$, the uniqueness of the Whittaker model implies that $W_\varphi = \bigotimes_v W_v$. Thus if $f_s = \bigotimes_v f_s^{(v)}$, then for $Re(s)$ large

$$I(\varphi,\chi,f_s,s) = \prod_v I_v(W_v,\chi_v,f_s^{(v)},s)$$

where

$$I_v(W_v,\chi_v,f_v^{(s)},s) = \int\limits_{Z(F_v)V(F_v)\backslash H(F_v)} \int\limits_{F_v} W_v\big(x_{-00001}(r)\,j(g_1,g_2)\big)\,f_s^{(v)}\,(g_1)\,dr\,dg_1\,dg_2\ .$$

A similar statement holds for $J(\varphi,\chi,f_s,s)$. The aim of the next section is to study these local integrals.

3. The local theory

Let F be a local field. When there is no confusion we shall write G for $G(F)$, etc. Let π be a generic irreducible admissible representation of GSp_{2n}. Let ψ be an additive character of F. We shall denote by $\mathcal{W}(\pi,\psi)$ the Whittaker model of π. Let ω_π be the central character of π and denote by χ a character of F^*. When $n=5$, we set $I(s,\chi_\pi) = \mathrm{Ind}_P^{GSp_6}\delta_P^s\chi_\pi$ (see Section 2 for notations). Thus $P = MR$ is the Siegel parabolic in GSp_6, i.e. $M = GL_1\times GL_3$. Also

$$\chi_\pi\big((\alpha,g)r\big) = (\omega_\pi\chi^3)(\alpha)(\omega_\pi\chi^2)(\det g)$$

for all $(\alpha,g)\in GL_1\times GL_3$ and $r\in R$. Thus in this section we shall study the local integral

$$I(W,\chi,f_s,s) = \int\limits_{ZV\backslash H}\int\limits_{F} W\big[x_{-00001}(r)j(g_1,g_2)\big]\,f_s(g_1)\,dr\,dg_1\,dg_2\ .$$

Similarly, when $n=4$ we let $I(s,\chi_\pi) = \mathrm{Ind}_B^{GL_2}\delta_B^s\chi_\pi$ where

$$\chi_\pi\begin{pmatrix} a & x\\ 0 & b\end{pmatrix} = \omega_\pi\chi(a)\chi^{-1}(b)\ .$$

The local integral in this case is

$$J(W,\chi,f_s,s) = \int\limits_{ZV\backslash H}\int\limits_{F^4} W\big(X(m_1,m_2,m_3,m_4)j(g_1,g_2)\big)f_s(g_1)dm_i\,dg_j\ .$$

If F is nonarchimedean, $\mathcal{O}$ will denote its ring of integers, p the generator of the maximal ideal in $\mathcal{O}$ and $q^{-1} = |p|$. For a given reductive group G, $K(G)$ will denote its standard maximal compact subgroup. Finally, for χ as above we set

$$L(\chi,s) = \big(1-\chi(p)q^{-s}\big)^{-1}\ .$$

3.1 The unramified computations.

a) The GSp_{10} case. Let F be nonarchimedean. In this section we assume that all data are unramified. Thus, W is the $K(GSp_{10})$ fixed vector with $W(e)=1$ and similarly f_s is $K(GSp_6)$ fixed with $f_s(e)=1$. We also assume that ψ and χ are unramified.

The L-group of GSp_{10} is $\mathrm{GSpin}_{11}(\mathbb{C})$. This group has a 32-dimensional irreducible analytic representation referred to as the Spin representation. If we use the notations of Brion [B] and denote by $\widetilde{\omega}_i$ for $1 \le i \le 5$ the fundamental representations of $\mathrm{GSpin}_{11}(\mathbb{C})$, then the Spin representation is $\widetilde{\omega}_5$. Given π as above we may associate with it a semisimple conjugacy class t_π of $\mathrm{GSpin}_{11}(\mathbb{C})$.

We define the local twisted Spin L-function of GSp_{10} as

$$L(\pi\otimes\chi,\mathrm{Spin},s)=\left\{\det\left(I-\mathrm{Spin}(t_\pi)\chi(p)q^{-s}\right)\right\}^{-1} .$$

Here I denotes the $32\times$ 32 identity matrix. In this section we prove

Proposition 3.1. *For all unramified data as above and for $Re(s)$ large, we have*

$$I(W,\chi,f_s,s)=\frac{L(\pi\otimes\chi,\mathrm{Spin},2s-1/2)}{L(\omega_\pi\chi^2,4s)\,L(\omega_\pi^2\chi^4,8s-2)} .$$

Proof. We start by writing the Iwasawa decomposition for H. Parameterize the maximal torus of $Z\backslash H$ as

$$t=(t_1,t_2)=\big(\mathrm{diag}(y_1y_2y_3y_4y_5,y_1y_2y_3,1,y_1,y_2^{-1}y_3^{-1},y_2^{-1}y_3^{-1}y_4^{-1}y_5^{-1}),\\ \mathrm{diag}\big(y_1y_2y_3y_4,y_1y_2,y_2^{-1},y_2^{-1}y_3^{-1}y_4^{-1}\big)\big) .$$

One can check that $j(t)$ equals

$$\mathrm{diag}(y_1y_2y_3y_4y_5,y_1y_2y_3y_4,y_1y_2y_3,y_1y_2,y_1,1,y_2^{-1},y_2^{-1}y_3^{-1},y_2^{-1}y_3^{-1}y_4^{-1},y_2^{-1}y_3^{-1}y_4^{-1}y_5^{-1}).$$

Let $B(G)$ denote the maximal standard Borel subgroup of G where $G=GSp_4$ or $G=GSp_6$. One easily checks that $\delta_{B(GSp_4)}(t_2)=|y_1^3y_2^6y_3^4y_4^4|$ and $\delta_{B(GSp_6)}(t_1)=|y_1^2y_2^8y_3^8y_4^4y_5^4|$. Also one has $\delta_P(t_1)=|y_1^2y_2^8y_3^8y_4^4y_5^4|$. Choosing the measure on $K(H)$ so that its volume is one we obtain

$$I(W,\chi,f_s,s)=\int\limits_{(F^*)^5}\int\limits_F W\big[x_{-00001}(r)j(t)\big]\chi(y_1y_2^4y_3^4y_4^2y_5^2)\\ \times\omega_\pi(y_2^2y_3^2y_4y_5)|y_1^2y_2^8y_3^8y_4^4y_5^4|^s|y_1^7y_2^{16}y_3^{14}y_4^{10}y_5^6|^{-1}\,dr\,d^\times y_i .$$

Conjugating the torus across the unipotent matrix and changing variables, this equals

$$\int\limits_{(F^*)^5}\int\limits_F W\big(j(t)x_{-00001}(r)\big)\chi(y_1y_2^4y_3^4y_4^2y_5^2)\omega_\pi(y_2^2y_3^2y_4y_5)\\ \times|y_1^2y_2^8y_3^8y_4^4y_5^4|^s|y_1^8y_2^{16}y_3^{14}y_4^{10}y_5^6|^{-1}\,dr\,d^\times y_i .$$

Next we split the integration domain on r into $|r|\le 1$ and $|r|>1$. For $|r|>1$ we use the identity $\begin{pmatrix}1&\\r&1\end{pmatrix}=\begin{pmatrix}r^{-1}&1\\&r\end{pmatrix}k$ with $k\in K(SL_2)$. Thus, for the domain

$|r| > 1$, we have

$$\int_{(F^*)^5} \int_{|r|>1} W\big(j(t)x_{00001}(r^{-1})h_5(r^{-1})\big)\chi(y_1y_2^4y_3^4y_4^2y_5^2)\omega_\pi(y_2^2y_3^2y_4y_5) \times |y_1^2y_2^8y_3^8y_4^4y_5^4|^s|y_1^8y_2^{16}y_3^{14}y_4^{10}y_5^6|^{-1}\,dr\,d^\times y_i\ .$$

Expressing $h_5(r^{-1})$ in terms of the maximal torus in GSp_{10}, and using the invariance properties of W, we get

$$W\big(j(t)x_{00001}(r^{-1})h_5(r^{-1})\big) = \omega_\pi(r)\,\psi(r^{-1}y)W\big(j(t)h(r^{-2})\,h(r,r,r,r,1)\big)\ .$$

Substituting this into the above integral and changing variables $y_1 \to y_1r^2$ and $y_2 \to y_2r^{-1}$ we obtain

$$\begin{aligned}&\int_{(F^*)^5} W\big(j(t)\big)\chi(y_1y_2^4y_3^4y_4^2y_5^2)\omega_\pi(y_2^2y_3^2y_4y_5)|y_1^2y_2^8y_3^8y_4^4y_5^4|^s \\ &\quad\times |y_1^8y_2^{16}y_3^{14}y_4^{10}y_5^6|^{-1}\left(\int_{|r|>1}\omega_\pi^{-1}(r)\chi^{-2}(r)\,|r|^{-4s}\,\psi(r^{-1}y_1)\,dr\right)d^\times y_i\ .\end{aligned} \tag{3.1}$$

We have $I(W,\chi,f_s,s) = \int_{(F^*)^5}\int_{|r|\le 1} + \int_{(F^*)^5}\int_{|r|>1}$. In the first integral on the right-hand side we may ignore the r integration (choosing the measure so that $\int_{|r|\le 1} dr = 1$). Combining this with (3.1) we obtain

$$\begin{aligned}I(W,\chi,f_s,s) = &\int_{(F^*)^5} W\big(j(t)\big)\chi(y_1y_2^4y_3^4y_4^2y_5^2)\omega_\pi(y_2^2y_3^2y_4y_5) \\ &\times |y_1^2y_2^8y_3^8y_4^4y_5^4|^s|y_1^8y_2^{16}y_3^{14}y_4^{10}y_5^6|^{-1}L(y_1)d^\times y_i\end{aligned} \tag{3.2}$$

where

$$L(y_1) = 1 + \int_{|r|>1}\omega_\pi^{-1}(r)\chi^{-2}(r)|r|^{-4s}\psi(r^{-1}y_1)dr\ .$$

It follows from the properties of W that we may restrict the domain of integration in (3.2) to $|y_i| \le 1$, $1 \le i \le 5$. It follows from [G1] that, for $|y_1| \le 1$,

$$L(y_1) = \frac{L(\omega_\pi\chi^2, 4s-1)}{L(\omega_\pi\chi^2, 4s)}\big(1 - \omega_\pi\chi^2(y_1)|y_1|^{4s-1}\omega_\pi\chi^2(p)q^{-4s+1}\big)\ .$$

Set $K\big(j(t)\big) = \delta_{B(GSp_{10})}^{-1/2}\big(j(t)\big)W\big(j(t)\big)$. We have

$$\delta_{B(GSp_{10})}^{-1/2}\big(j(t)\big) = |y_1^{15/2}\,y_2^{14}\,y_3^{12}\,y_4^9\,y_5^5|.$$

Write $y_i = p^{n_i}\epsilon_i$ with $n_i \ge 0$ and $|\epsilon_i| = 1$. Thus, normalizing the multiplicative measure so that $\int_{\mathcal{O}^*} d\epsilon_i = 1$, we have

$$\begin{aligned}I(W,\chi,f_s,s) = &\frac{L(\omega_\pi\chi^2, 4s-1)}{L(\omega_\pi\chi^2, 4s)}\sum_{n_i=0}^{\infty} K\big(j(t)\big)\,\chi(p)^{n_1+4n_2+4n_3+2n_4+2n_5} \\ &\times \omega_\pi(p)^{2n_2+2n_3+n_4+n_5}\,q^{(-2s+1/2)n_1+(-8s+2)n_2+(-8s+2)n_3+(-4s+1)n_4+(-4s+1)n_5} \\ &\times \big(1 - \omega_\pi\chi^2(p)^{n_1+1}q^{(-4s+1)(n_1+1)}\big),\end{aligned}$$

where

$$j(t) = \mathrm{diag}(p^{n_1+n_2+n_3+n_4+n_5}, p^{n_1+n_2+n_3+n_4}, p^{n_1+n_2+n_3}, p^{n_1+n_2}, p^{n_1}, 1, p^{-n_2}, p^{-n_2-n_3}, p^{-n_2-n_3-n_4}, p^{-n_2-n_3-n_4-n_5}).$$

On the other hand using the Poincaré identity, we have

$$L(\pi \otimes \chi, \mathrm{Spin}, 2s-1/2) = \sum_{n=0}^{\infty} \mathrm{tr} S^n(t_\pi)\chi(p)^n q^{(-2s+1/2)n}$$

where S^n denotes the symmetric n-th power operation. Thus to prove the proposition we need to show that

$$\left(1-\omega_\pi(p)x^2\right)\left(1-\omega_\pi^2(p)x^4\right)\sum_{n=0}^{\infty} \mathrm{tr} S^n(t_\pi)x^n = \sum_{n_i=0}^{\infty} K\big(j(t)\big)\,\omega_\pi(p)^{2n_2+2n_3+n_4+n_5} \times x^{n_1+4n_2+4n_3+2n_4+2n_5}\left(1-\omega_\pi(p)^{n_1+1}\,x^{2(n_1+1)}\right).$$

Denote by $(0\cdots 0,1,0\cdots 0)$, one in the i-th position and zero elsewhere, the character of the representation $\widetilde{\omega}_i$ evaluated at t_π. The Casselman-Shalika formula [C-S] states that $K\big(j(t)\big) = (n_5, n_4, n_3, n_2, n_1)$.

Thus to prove the proposition we need to show that

(3.3)
$$\sum_{n=0}^{\infty} \mathrm{tr}\ S^n(t_\pi)x^n = \frac{1}{1-\omega_\pi^2(p)x^4} \sum_{\substack{n_i=0\\ 1\le i\le 5}} (n_5,n_4,n_3,n_2,n_1)\,\omega_\pi(p)^{2n_2+2n_3+n_4+n_5} \times x^{n_1+4n_2+4n_3+2n_4+2n_5}\left(\frac{1-\omega_\pi(p)^{n_1+1}\,x^{2(n_1+1)}}{1-\omega_\pi(p)\,x^2}\right).$$

Next we use Brion's result in [B] to decompose the Symmetric algebra of the Spin representation. More precisely, let $\widetilde{V}$ be the 32-dimensional irreducible complex Spin representation of GSpin_{11}. Let $\widetilde{U}$ denote the maximal unipotent subgroup of GSpin_{11}. It follows from the table on page 13 in [B] that

$$\text{(3.4)}\quad \mathrm{tr}\ S^r(t_\pi) = \sum (m_2, m_3, m_4, m_5, m_1+m_6)\omega_\pi(p)^{m_2+m_3+2m_4+2m_5+m_6+4m_7}$$

where the summation is over all m_i where $1 \le i \le 7$ satisfying $m_1 + 2m_2 + 2m_3 + 4m_4 + 4m_5 + 3m_6 + 4m_7 = r$. Indeed this follows from the fact that the ring $\mathbb{C}[\widetilde{V}]^{\widetilde{U}}$ of $\widetilde{U}$ invariants of the symmetric algebra is free and generated by $(1,\widetilde{\omega}_5); (2,\tilde{\omega}_1); (2,\tilde{\omega}_2); (3,\tilde{\omega}_5); (4,\tilde{\omega}_3); (4,\tilde{\omega}_4)$ and (4,0) (see [B] for notations). Also the extra power of $\omega_\pi(p)$ in the above identity follows from the fact that t_π is in $\mathrm{GSpin}_{11}(\mathbb{C})$ and not necessarily in $\mathrm{Spin}_{11}(\mathbb{C})$ as in [G1]. Multiply (3.4) by x^r and sum over all r. We see that (3.4) equals

$$\sum_{r=0}^{\infty}\sum_{\substack{m_i=0\\ 1\le i\le 7}}^{\infty} (m_2,m_3,m_4,m_5,m_1+m_6)\,\omega_\pi(p)^{m_2+m_3+2m_4+2m_5+m_6+2m_7} \times x^{m_1+2m_2+2m_3+4m_4+4m_5+3m_6+4m_7}.$$

Using the geometric series formula we see that the summation over m_7 is $(1-\omega_\pi^2(p)x^4)^{-1}$. Thus

$$\sum_{r=0}^{\infty} \text{tr } S^r(t_\pi)x^r = \left(1-\omega_\pi^2(p)x^4\right)^{-1} \sum_{\substack{m_i=0\\1\le i\le 6}}^{\infty} (m_2,m_3,m_4,m_5,m_1+m_6) \times \omega_\pi(p)^{m_2+m_3+2m_4+2m_5+m_6} x^{m_1+2m_2+2m_3+4m_4+4m_5+3m_6} .$$

Set on the right-hand side $m_1+m_6=\ell$. We get

$$\sum_{r=0}^{\infty} trS^r(t_\pi)x^r = \left(1-\omega_\pi^2(p)x^4\right)^{-1} \sum_{\substack{m_i=0\\2\le i\le 6}}^{\infty} \sum_{\ell=m_6}^{\infty} (m_2,m_3,m_4,m_5,\ell) \times \omega_\pi(p)^{m_2+m_3+2m_4+2m_5+m_6} x^{\ell+2m_2+2m_3+4m_4+4m_5+2m_6} . \tag{3.5}$$

The right side of (3.3) equals

$$\begin{aligned}
&\left(1-\omega_\pi^2(p)x^4\right)^{-1} \sum_{\substack{n_i=0\\1\le i\le 5}}^{\infty} (n_5,n_4,n_3,n_2,n_1)\omega_\pi(p)^{2n_2+2n_3+n_4+n_5} \\
&\qquad \times x^{n_1+4n_2+4n_3+2n_4+2n_5}\left(1+\omega_\pi(p)x^2+\cdots+\omega_\pi(p)^{n_1}x^{2n_1}\right) \\
&= \left(1-\omega_\pi^2(p)x^4\right)^{-1} \sum_{\substack{n_i=0\\1\le i\le 5}} \sum_{\ell=0}^{n_1} (n_5,n_4,n_3,n_2,n_1)\omega_\pi(p)^{\ell+2n_2+2n_3+n_4+n_5} \\
&\qquad \times x^{2\ell+n_1+4n_2+4n_3+2n_4+2n_5} .
\end{aligned}$$

Interchange the summation of n_1 and ℓ to get

$$\left(1-\omega_\pi^2(p)x^4\right)^{-1} \sum_{\substack{\ell,n_i=0\\1\le 2\le 5}}^{\infty} \sum_{n_1=\ell}^{\infty} (n_5,n_4,n_3,n_2,n_1) \times \omega_\pi(p)^{\ell+2n_2+2n_3+n_4+n_5} x^{2\ell+n_1+4n_2+4n_3+2n_4+2n_5} .$$

Now (3.3) follows immediately from (3.5). This completes the proof of Proposition 3.1. □

b) The GSp_8 case. As in the GSp_{10} case we assume that all data is unramified. We denote by

$$L(\pi\otimes\chi,\text{Spin},s) = \left\{\det\left(I-\text{Spin}(t_\pi)\chi(p)q^{-s}\right)\right\}^{-1}$$

the local Spin L-function of $\text{GSpin}_9(\mathbb{C})$. This L-function is of degree 16. We have

Proposition 3.2. *For all unramified data and for Re(s) large*

$$J(W,\chi,f_s,s) = \frac{L(\pi\otimes\chi,\text{Spin},s)}{L(\omega_\pi\chi^2,2s)} .$$

Proof. Parameterize the maximal torus of $Z\backslash H$ as $t = \left(\begin{pmatrix} ab & \\ & b^{-1}\end{pmatrix}, \begin{pmatrix} a & \\ & 1\end{pmatrix}\right)$. Then

$$j(t) = \text{diag}(ab,a,a,a,1,1,1,b^{-1}) .$$

Applying the Iwasawa decomposition we obtain

$$J(W,\chi,f_s,s) = \int_{(F^*)^2}\int_{F^4} W\big(X(m_1,m_2,m_3,m_4)j(t)\big)\omega_\pi\chi(a)\chi^2(b)|ab^2|^{s-1}|a|^{-1}dm_i d^\times a d^\times b\ .$$

Conjugating $j(t)$ across the unipotent element we obtain after a change of variables

$$J(W,\chi,f_s,s) = \int_{(F^*)^2}\int_{F^4} W\big(j(t)X(m_1,m_2,m_3,m_4)\big)\omega_\pi(a)\chi(ab^2)|a|^{s-5}|b|^{2s-4}dm_i d^\times a d^\times b\ .$$

Define

$$F(m_1) = \int_{F^3} W\big(j(t)X(0,m_2,m_3,m_4)X(m_1,0,0,0)\big)dm_2 dm_3 dm_4\ .$$

Since W is $K(GSp_8)$ fixed, it is invariant on the right by $x_{1100}(r)$ for $|r|\le 1$. Thus

$$F(m_1) = \int_{F^3} W\big(j(t)\,X(0,m_2,m_3,m_4)\,X(m_1,0,0,0)\,x_{1100}(r)\,\big)\,dm_2\,dm_3\,dm_4$$
$$= \psi(m_1 r)\int_{F^3} W\big(j(t)\,X(0,m_2,m_3,m_4)\,X(m_1,0,0,0)\big)\,dm_2\,dm_3\,dm_4$$

where the last equality is obtained by conjugating $x_{1100}(r)$ to the left and using the left transformation properties of W. Hence $F(m_1)=\psi(m_1r)\,F(m_1)$. This implies that $F(m_1)=0$ if $|m_1|>1$. Thus

$$J(W,\chi,f_s,s) = \int_{(F^*)^2}\int_{F^3} W\big(j(t)X(0,m_2,m_3,m_4)\big)\,\omega_\pi(a)\,\chi(ab^2)\,|a|^{s-4}\,|b|^{2s-4}\,dm_i\,d^\times a\,d^\times b\ .$$

Repeating this process with $x_{1111}(r)$ for m_2, $x_{0111}(r)$ for m_3 and $x_{0011}(r)$ for m_4 we obtain

$$J(W,\chi,f_s,s) = \int_{(F^*)^2} W\big(j(t)\big)\,\omega_\pi(a)\,\chi(ab^2)\,|a|^{s-5}\,|b|^{2s-4}\,d^\times a\,d^\times b\ .$$

Set $W\big(j(t)\big)=\delta^{1/2}_{B(GSp_8)}\big(j(t)\big)\,K\big(j(t)\big)$, where $B(GSp_8)$ is the standard Borel subgroup of GSp_8 containing U_4. Thus $\delta^{1/2}_{B(GSp_8)}\big(j(t)\big)=|a^{-3}b^{-2}|$. Hence

$$J(W,\chi,f_s,s) = \int_{(F^*)^2} K\big(j(t)\big)\omega_\pi(a)\chi(ab^2)|ab^2|^s d^\times a d^\times b.$$

We know that $K\big(j(t)\big)=0$ if $|a|>1$ or $|b|>1$. Thus

$$J(W,\chi,f_s,s) = \sum_{n,m=0}^{\infty} K\big(d(p^n,p^m)\big)\omega_\pi(p)^n x^{n+2m}$$

where $d(p^n,p^m)=\mathrm{diag}(p^{n+m},p^n,ap^n,p^n,1,1,1,p^{-m})$ and $x=\chi(p)\,q^{-s}$.

Finally we use the result of Brion [B] in a similar way as in the GSp_{10} case. We omit the details. □

3.2 Some nonvanishing results. In this section we shall prove some nonvanishing results. We shall concentrate on the GSp_{10} case. The GSp_8 case is done similarly (see also the E_6 case in [G2]). To further study our local integrals we need the following asymptotic expansion of the Whittaker functions.

Lemma 3.3. *There is a finite set X of finite functions of $(F^*)^5$, such that for all $W \in \mathcal{W}(\pi,\psi)$ and for $\alpha \in X$ there is $\phi_\alpha \in \mathcal{S}\big(F^5 \times K(GSp_{10})\big)$ such that*

$$W\big(j(t)k\big) = \sum_{\alpha \in X} \phi_\alpha(y_1,y_2,y_3,y_4,y_5,k)\,\alpha(y_1,y_2,y_3,y_4,y_5)$$

where $j(t)$ is parameterized as in the proof of Proposition 3.1 and $k \in K(GSp_{10})$.

Proof. The proof follows as in Jacquet and Shalika [J-S] or Soudry [S] Sections 2,3. □

Using this we prove:

Lemma 3.4. *The integral $I(W,\chi,f_s,s)$ converges absolutely for $Re(s)$ large.*

Proof. Proceeding as in the first steps in Proposition 3.1 we obtain for $Re(s)$ large

$$I(W,\chi,f_s,s) = \int\limits_{K(H)} \int\limits_{(F^*)^5} \int\limits_{F} W\big(j(t)x_{-00001}(r)\,(k_1,k_2)\big)\, f_s(k_1)\,\mu(y_1,y_2,y_3,y_4,y_5) \times |y_1^2y_2^8y_3^8y_4^4y_5^4|^s\,dr\,d^\times t\,dk_1\,dk_2$$

where μ is a character in y_i which depends on ω_π,χ and on $|y_i|$ for $1 \le i \le 5$. Next we write the Iwasawa decomposition for $x_{-00001}(r)$. In GL_2 we have for F nonarchimedean

$$\begin{pmatrix} 1 & \\ r & 1 \end{pmatrix} = \begin{pmatrix} -r^{-1} & 1 \\ & r \end{pmatrix} k_r \qquad |r| > 1$$

and if F is archimedean

$$\begin{pmatrix} 1 & \\ r & 1 \end{pmatrix} = \begin{pmatrix} (1+|r|^2)^{-1/2} & * \\ & (1+|r|^2)^{1/2} \end{pmatrix} k'_r, \qquad r \neq 0,$$

where k_r, k'_r are in $K(GL_2)$. Thus we may write $x_{-00001}(r) = uh_5(z)k_r$ where $u \in U$ and $z = r^{-1}$ if F is nonarchimedean and $z = (1+r^2)^{-1/2}$ if F is archimedean. In any case $|z| \le 1$. Thus we need to consider the convergence of

$$\int\limits_{K(H)} \int\limits_{(F^*)^5} \int\limits_{F} \Big| W\big(j(t)h_5(z)k_r(k_1,k_2)\big)\Big| \times |f_s(k_1)|\,|\mu(y_1,\cdots,y_5)|\,|y_1^2y_2^8y_3^8y_4^4y_5^4|^s\,dr\,d^\times t\,dk_1\,dk_2\ .$$

Changing variables $y_1 \to y_1z^{-2}$ and $y_2 \to y_2z$ we obtain

$$\int\limits_{K(H)} \int\limits_{(F^*)5} \int\limits_{F} \Big| W\big(j(t)k_r(k_1,k_2)\big)\Big| \times\ |f_s(k_1)|\,|\mu(y_1,\cdots,y_5)|\,|y_1^2y_2^8y_3^8y_4^4y_5^4|^s|z|^{4s}\,dr\,d^\times t\,dk_1\,dk_2\ .$$

Using Lemma 3.2 we can bound the above integral by

$$\int\limits_{(F^*)^5}\int\limits_F (\nu\phi)(y_1,y_2,y_3,y_4,y_5)|\mu(y_1,\cdots,y_5)|\ |y_1^2y_2^8y_3^8y_4^4y_5^4|^s|z|^{4s}\,dr\,d^\times t$$

where ν is a finite function and ϕ a positive valued Schwartz-Bruhat function. This integral clearly converges for $Re(s)$ large. (Recall that as r varies in F, then $|z|\le 1$.) □

Next we prove

Lemma 3.5. *Let f_s be a standard $K(GSp_6)$ finite section. Then $I(W,\chi,f_s,s)$ admits a meromorphic continuation to the whole complex plane, which is also continuous in W.*

Proof. Let $x_\gamma(r)$ denote the one parameter unipotent subgroup of GSp_6 defined by

$$x_\gamma(r)=\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&r&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix},\quad r\in F,$$

and denote by w_γ the simple reflection in GSp_6 corresponding to the root γ. It is not difficult to check that

$$j\big((x_\gamma(r),1)\big)=x_{-00001}(r)\ .$$

Thus we may rewrite our integral as

$$I(W,\chi,f_s,s)=\int\limits_{ZV\backslash H}\tilde W\big(j'(g_1,g_2)\big)\left(\int\limits_F f_s\big(w_\gamma x_\gamma(r)g_1\big)\,\psi(r)\,dr\right)dg_1dg_2$$

where $j'(g_1,g_2)=j\big((w_\gamma,1)(g_1,g_2)(w_\gamma,1)^{-1}\big)$ and $\tilde W=\Big(j\big((w_\gamma,1)\big)\Big)W$, i.e. $\tilde W$ is the right translate of W by $j\big((w_\gamma,1)\big)$.

Since f_s is $K(GSp_6)$ finite, then replacing $\tilde W$ by W, we need to study the meromorphic continuation of

$$\int\limits_{(F^*)^5}W\big(j(t)\big)\left(\int\limits_F f_s\big(w_\gamma x_\gamma(r)\big)\,\psi(y_1r)\,dr\right)\mu_s(t)\,d^\times t$$

where $j(t)$ is as in the proof of Proposition 3.1 and $\mu_s(t)$ is a function of t which depends on χ,ω_π and the absolute value of y_i for $1\le i\le 5$. Thus we are reduced to a similar situation as in (4.7) in [G1] Lemma 4.3. Proceeding the same way we prove the lemma. □

Lemma 3.6. *Assume f_s is $K(GSp_6)$ finite. Given $s_0\in\mathbb{C}$ there is a choice of data such that $I(W,\chi,f_s,s)$ is nonzero at s_0.*

Proof. Let $X(r_2, r_3, r_4) = x_{01100}(r_2)x_{00100}(r_3)x_{00110}(r_4)$. Also define $L \subset H$ by

$$L = \left(\begin{pmatrix} 1 & z_1 & z_2 & & & \\ & 1 & z_4 & & & \\ & & 1 & & & \\ & & & 1 & * & * \\ & & & & 1 & * \\ & & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & z_1 & z_2 & z_3 \\ & 1 & z_4 & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right)$$

where $*$ indicates that the above matrices are in H. We have for $Re(s)$ large

(3.6)

$$I(W, \chi, f_s, s) = \int_{ZL\backslash H} \int_{F^4} W\big(x_{-00001}(r_1)j\big(X(r_2, r_3, r_4)j(g_1, g_2)\big)\, f_s(g_1)\, dr_i\, dg_1\, dg_2\ .$$

Indeed, factoring the measure $\int\limits_{ZL\backslash H} = \int\limits_{ZV\backslash H} \int\limits_{L\backslash V}$ we may identify the quotient $L\backslash V$ with the subgroup of H

$$\left(\begin{pmatrix} 1 & z_1 & z_2 & & & \\ & 1 & z_4 & & & \\ & & 1 & & & \\ & & & 1 & * & * \\ & & & & 1 & * \\ & & & & & 1 \end{pmatrix}, 1 \right) = x_{10000}(z_1)x_{11000}(z_2)x_{01000}(z_4)\ .$$

Thus the right-hand side of (3.6) becomes

$$\int_{ZV\backslash H} \int_{F^3} \int_{F^4} W\big(x_{-00001}(r_1)j\big(X(r_2, r_3, r_4)x_{10000}(z_1)x_{11000}(z_2)z_{01000}(z_4)(g_1, g_2)\big)$$
$$\times f_s(g_1)\, dr_i\, dz_k\, dg_1\, dg_2$$
$$= \int_{ZV\backslash H} \int_{F^7} W\Big(x_{-00001}(r_1)j\big(X(r_2, r_3, r_4)(g_1, g_2)\big)\Big)\psi(z_1r_2 + z_2r_3 + z_3r_4)$$
$$\times f_s(g_1)\, dr_i\, dz_k\, dg_1\, dg_2\ .$$

Using the properties of the Fourier transform of smooth functions (3.6) follows.

For this lemma only we shall write $I(W, \chi, f_s, s)$ for the right-hand side of (3.6). We shall prove that $I(W, \chi, f_s, s)$ is nonzero for $s = s_0$ for some choice of data. Define $I_1(W, \chi, s, k)$ to be

$$\int_{ZL\backslash(GL_1\times GL_3, GSp_4)} \int_{F^4} W\big(x_{-00001}(r_1)j\big(X(r_2, r_3, r_4)\big((\alpha, g_1), g_2\big)(k, 1)\big)$$
$$\times \mu_s(\alpha, g_1)\, dr_i\, dg_1\, dg_2\, d^\times\alpha$$

where $k \in K(GSp_6)$ and $\mu_s(\alpha, g_1)$ is a function which depends on the absolute value of the similitude factor of $GL_1 \times GL_3$ and on the determinant of the GL_3 part. It also depends on χ and ω_π. It follows from Lemma 3.4 that $I_1(W, \chi, s, k)$ admits a meromorphic continuation to the whole complex plane and defines a continuous function of k. From the assumption that $I(W, \chi, f_{s_0}, s_0) = 0$ for any choice of data it follows that

$$\int\limits_{K(GSp_6)\cap(GL_1\times GL_3)\backslash K(GSp_6)} I_1(W,\chi,s,k)\sigma(k)\,dk$$

is zero for $s=s_0$ and for all Schwartz-Bruhat functions σ on $K(GSp_6)\cap(GL_1\times GL_3)$. Thus $I_1(W,\chi,s,k)$ equals zero at $s=s_0$ for all W and all k. Choose $k=e$.

Recall that the unipotent radical of the parabolic subgroup of GSp_{10} whose Levi part is $GL_3\times GSp_4$ is a two step unipotent subgroup. Denote by T the quotient of this radical modulo its center. Thus T may be identified with $M_{3\times 4}$. Replace W in $I_1(W,\chi,s,e)$ by

$$W_1(m)=\int\limits_T \phi(t)W(mt)dt,\qquad m\in GSp_{10},$$

where $\phi\in S(M_{3\times 4})$. Here we view T as $M_{3\times 4}$ and as embedded in GSp_{10}. Notice that $(GL_1\times GL_3, GSp_4)$ acts on T as $(\alpha,g_1)tg_2^{-1}$ when T is identified with $M_{3\times 4}$. Thus, for re(s) large, $I_1(W_1,\chi,s,e)$ equals

$$\int\limits_{ZL\backslash(GL_1\times GL_3,GSp_4)}\ \int\limits_{F^4}\int\limits_T W_1\Big(x_{-00001}(r_1)j\big(X(r_2,r_3,r_4)\big((\alpha,g_1),g_2)t\big)\Big)$$
$$\times\,\phi(t)\,\mu_s(\alpha,g_1)\,dt\,dr_i\,dg_1\,dg_2\,d^\times\alpha$$
$$=\int\int\limits_{F^4} W_1\Big(x_{-00001}(r_1)j(X(r_2,r_3,r_4))\big((\alpha,g_1),g_2\big)\Big)\widehat{\phi}\big((\alpha,g_1)x_5g_2^{-1}\big)$$
$$\times\,\mu_s\,(\alpha,g_1)dr_i\,dg_1\,dg_2\,d^\times\alpha\ .$$

The last equality is obtained by conjugating t to the left in W_1. $\widehat{\phi}$ is the Fourier transform of ϕ and x_5 is as defined in the proof of Theorem 2.1. Arguing as before we obtain that

$$\int\limits_{\mathrm{Stab}(x_5)\backslash(GL_1\times GL_3,GSp_4)} I_2\Big(W,\chi,s,\big((\alpha,g_1),g_2\big)\Big)\sigma\big((\alpha,g_1)x_5g_2^{-1}\big)d^\times\alpha dg_1\,dg_2$$

is zero at $s=s_0$ for any choice of data and all Schwartz-Bruhat functions on T. Here

$$I_2\Big(W,\chi,s,\big((\alpha,g_1),g_2\big)\Big)=b_s(\alpha,g_1)\int\limits_{ZL\cap\mathrm{Stab}(x_5)\backslash\mathrm{Stab}(x_5)}$$
$$\times\int_{F^4}W\Big[x_{-00001}(r_1)j\big(X(r_2,r_3,r_4)m\big)j\big((\alpha,g_1),g_2\big)\Big]\,a_s(m)\,dr_i\,dm\ .$$

Here $b_s(\alpha,g_1)$ and $a_s(m)$ are certain functions of s which depend on χ,ω_π and the absolute values of the determinants of the arguments. Thus we obtain that $I_2(W,\chi,s,e)$ is zero at $s=s_0$ for all W. From (2.2) we may identify the quotient $ZL\cap\mathrm{Stab}(x_5)\backslash\mathrm{Stab}(x_5)$ with $N\backslash GL_2^\Delta$ where GL_2^Δ is embedded in H as

$$\left(\begin{pmatrix}|g| &&&\\ & g &&\\ && g^* &\\ &&& 1\end{pmatrix},\begin{pmatrix}|g| &&\\ & g &\\ && 1\end{pmatrix}\right)$$

and N is the group of upper triangular unipotent matrices in GL_2^Δ. Thus we need to study

$$I_2(W,\chi,s,e)=\int\limits_{N\backslash GL_2^\Delta}\ \int\limits_{F^4} W\Big[x_{-00001}(r_1)j\big(X(r_2,r_3,r_4)m\big)\Big]\,a_s(m)\,dr_i\,dm\ .$$

We may conjugate $X(r_2,r_3,0)$ across m (this entails a change of variables). Thus $I_2(W,\chi,s,e)$ equals

$$\int\limits_{N\backslash GL_2^\Delta}\ \int\limits_{F^4} W\Big[x_{-00001}(r_1)j(X(0,0,r_4)m)X(r_2,r_3,0)\Big]\,a_s(m)\,|m|^{-1}\,dr_i\,dm\ .$$

Replace W by

$$W_1(y)=\int\limits_{F^2}\phi(\ell_1,\ell_2)W(yj(x_{10000}(\ell_1)x_{11000}(\ell_2)))d\ell_1 d\ell_2$$

where $y\in GSp_{10}$ and $\phi\in\mathcal{S}(F^2)$. We thus obtain

$$I_2(W_1,\chi,s,e)=\int\limits_{N\backslash GL_2^\Delta}\ \int\limits_{F^4} W_1\Big[x_{-00001}(r_1)j(X(0,0,r_4)mX(r_2,r_3,0))\Big]$$
$$\times\,\widehat{\phi}(r_2,r_3)\,a_s(m)\,|m|^{-1}\,dr_i\,dm\ .$$

Once again we obtain that the meromorphic continuation of the above integral vanishes for all data at $s=s_0$ and hence the meromorphic continuation of

$$I_3(W,\chi,s)=\int\limits_{N\backslash GL_2^\Delta}\ \int\limits_{F^2} W\Big[x_{-00001}(r_1)j\big(X(0,0,r_4)m\big)\Big]\,a_s(m)\,|m|^{-1}\,dr_i\,dm$$

is zero at $s=s_0$. Continuing this process with the roots $j\big(x_{-01100}(\ell_1)x_{-00100}(\ell_2)\big)$ we obtain that the meromorphic continuation of

$$I_4(W,\chi,s)=\int\limits_{F^*}\ \int\limits_{F^2} W\Big[x_{-00001}(r_1)j\big(X(0,0,r_4)\alpha\big)\Big]\,c_s(\alpha)\,dr_i\,d^\times\alpha$$

where α is embedded in H as

$$\left(\begin{pmatrix}\alpha&&&&&\\&\alpha&&&&\\&&1&&&\\&&&\alpha&&\\&&&&1&\\&&&&&1\end{pmatrix},\begin{pmatrix}\alpha&&&\\&\alpha&&\\&&1&\\&&&1\end{pmatrix}\right).$$

Next, using $j\big(x_{01000}(\ell)\big)$ for the variable r_4 and $j\big(x_{-00110}(\ell)\big)$ for r_1 we obtain that the meromorphic continuation of

$$\int\limits_{F^*} W\big(j(\alpha)\big)d_s(\alpha)d^\times\alpha$$

is zero at $s=s_0$, for all W. Finally, using $x_{00001}(\ell)$, we obtain that $W(e)=0$ for all W. This is a contradiction. $\square$

4. The analytic properties of the partial Spin L-function

In this section we shall study the poles of the partial Spin L-function. Let $\pi = \bigotimes_v \pi_v$ be a generic cusp form on $GSp_{10}(\mathbb{A})$. Let S be a finite set of places including the infinite ones such that outside S all data are unramified. Thus if $W = \bigotimes_v W_v$, $f_s = \bigotimes_v f_s^{(v)}$ and $\chi = \bigotimes_v \chi_v$ then outside of S the functions W_v, $f_s^{(v)}$ and χ_v are all unramified. Define

$$L_S(\pi \otimes \chi, \mathrm{Spin}, s) = \prod_{v \notin S} L_v(\pi_v \otimes \chi_v, \mathrm{Spin}, s)$$

where the local L-functions are as defined in Section 3.1.

Given a character $\mu = \bigotimes_v \mu_v$ of $F^* \backslash \mathbb{A}^*$ set

$$L_S(\mu, s) = \prod_{v \notin S} L_v(\mu_v, s)$$

where $L_v(\mu_v, s) = (1 - \mu_v(p_v) q_v^{-s})^{-1}$ and p_v is a generator of the maximal ideal in the ring of integers of F_v. Also $q_v^{-1} = |p_v|$.

As in [G1] Section 5 we normalize our Eisenstein series and the global integral. Set

$$E^*(g, f_s, \chi, s) = L_S(\omega_\pi \chi^2, 4s) L_S(\omega_\pi^2 \chi^4, 8s-2) E(g, f_s, \chi, s)$$

and

$$I^*(\varphi, \chi, f_s, s) = L_S(\omega_\pi \chi^2, 4s) L_S(\omega_\pi^2 \chi^4, 8s-2) I(\varphi, \chi, f_s, s)\ .$$

The main proposition in this section is

Proposition 4.1. *Let $f_s \in I(s, \chi_\pi)$ be a standard section. Then*
a) If $\omega_\pi \chi^2 = 1$ or $\omega_\pi^2 \chi^4 \neq 1$, then $I^*(\varphi, \chi, f_s, s)$ is entire.
b) If $\omega_\pi^2 \chi^4 = 1$ but $\omega_\pi \chi^2 \neq 1$, then $I^*(\varphi, \chi, f_s, s)$ has at most a simple pole at $s = 1/4$ or $s = 3/4$.

Proof. It follows from Theorem 1.1 in [K-R] and Proposition 1.6 in [I] (see also Lemma 5.4 in [G1]) that for $Re(s) \geq 1/2$

0) If $\omega_\pi^2 \chi^4 \neq 1$, then $E^*(g, f_s, \chi, s)$ is entire
0) If $\omega_\pi^2 \chi^4 = 1$ but $\omega_\pi \chi^2 \neq 1$, then $E^*(g, f_s, \chi, s)$ has at most a simple pole at $s = 3/4$.
0) If $\omega_\pi \chi^2 = 1$, then $E^*(g, f_s, \chi, s)$ has at most a simple pole at $s = 1$ and $s = 3/4$.

The residue at the above points is studied in Corollary 6.3 in [K-R] and Proposition 1.10 in Ikeda [I]. Using the notation of [G1] Lemma 5.5 we have

0) If E^* has a pole at $s = 1$, then the residue is a character of the similitude of $GSp_6(\mathbb{A})$.
0) If $\omega_\pi \chi^2 = 1$, then the residue of E^* at $s = 3/4$ is proportional to $E(g, \tilde{f}, s_1)$ for some $\tilde{f}$ and s_1 (see Lemma 5.5 in [G1] for notation).

To prove the proposition we need only to consider the points $s = 3/4$ and $s = 1$. If $s = 1$ is a pole for E^*, then

$$\operatorname*{Res}_{s=1} I^*(\varphi, \chi, f_s, s) = c \int\limits_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \varphi\big(j(g_1, g_2)\big) \nu\big(\mu_3(g_1)\big)\, dg_1\, dg_2$$

where c is a constant. It follows from the theorem in [A-G-R] case (4) that the right-hand side is zero. Hence there is no pole at $s = 1$. Next assume $\omega_\pi \chi^2 = 1$ and $s = 3/4$. Using statement 2) about the residue of E^* at $s = 3/4$ we obtain

$$\operatorname*{Res}_{s=3/4} I^*(\varphi, \chi, f_s, s) = c \int\limits_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \varphi\big(j(g_1, g_2)\big)\, E(g_1, \tilde{f}, s_1)\, dg_1\, dg_2 \ .$$

It follows from [A-G-R] formula (3.12) that this integral is zero. This completes the proof of the proposition. □

Finally we have

Theorem 4.2. *Let π be a generic cusp form for $GSp_{10}(\mathbb{A})$. Let S be as before. Then*

$$L_S(\pi \otimes \chi, \mathrm{Spin}, s) = \prod_{v \notin S} L_v(\pi_v \otimes \chi_v, \mathrm{Spin}, s)$$

is entire unless $\omega_\pi^2 \chi^4 = 1$ and $\omega_\pi \chi^2 \neq 1$. In this case the above L-function can have at most a simple pole at $s = 1$ or $s = 0$.

Proof. We have

$$I^*(\varphi, \chi, f_s, s) = \prod_{v \in S} I_v(W_v, \chi_v, f_s^{(v)}, s) L_S(\pi \otimes \chi, \mathrm{Spin}, 2s - 1/2) \ .$$

By Lemma 3.5, if $v \in S$, we may choose the data so that $I_v(W_v, \chi_v, f_s^{(v)}, s)$ is non zero. Thus the theorem follows from Proposition 4.1. □

Similarly for the GSp_8. It is known that the normalizing factor for $E(g, f_s, \chi, s)$ is $L_S(\omega_\pi \chi^2, 2s)$. Define

$$E^*(g, f_s, \chi, s) = L_S(\omega_\pi \chi^2, 2s)\, E(g, f_s, \chi, s)$$

and

$$J^*(\varphi, \chi, f_s, s) = L_S(\omega_\pi \chi^2, 2s)\, J(\varphi, \chi, f_s, s) \ .$$

Arguing as is the GSp_{10} case we obtain

Theorem 4.3. *Let π be a generic cusp form for $GSp_8(\mathbb{A})$. Then $L_S(\pi \otimes \chi, \mathrm{Spin}, s)$ is entire unless $\omega_\pi \chi^2 = 1$. In this case the above partial L-function can have at most a simple pole at $s = 0$ or $s = 1$.*

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